ASCHAM SCHOOL

# YEAR 12
# MATHEMATICS EXTENSION 1
# TRIAL EXAMINATION 2015

GENERAL INSTRUCTIONS

5 minutes reading time.
Working time: 2 hours.
Use black or blue pen.
A table of standard integrals is provided separately.
Approved calculators and templates may be used.

Total Marks - 70

Section A – Multiple Choice (1 mark each)

Attempt Questions 1 to 10.
Allow approximately 15 minutes.
Select answers on the separate multiple choice sheet provided.
Write your BOS number on the multiple choice sheet.

Section B – Questions 11 – 14 (15 marks each)

Allow 1 hour 45 minutes.
Start each question in a new booklet.
If you use a second booklet for a question, place it inside the first.
Label extra booklets for the same question as, for example, Q11-2 etc.
Write your BOS number and question number on each booklet.

Blank page

**Section A - Multiple choice (10 marks)**
(Mark the correct answer on the sheet provided.)

1. The interval $AB$ between $A(2,-1)$ and $B(-6,3)$ is divided internally by the point $P$ in the ratio $1:3$. The correct coordinate of $P$ is given by:

A) $\left(\frac{1\times 2+3\times -6}{1+3},\frac{1\times -1+3\times 3}{1+3}\right)$ B) $\left(\frac{1\times 2-3\times -6}{1-3},\frac{1\times -1-3\times 3}{1-3}\right)$

C) $\left(\frac{1\times -6+3\times 2}{1+3},\frac{1\times 3+3\times -1}{1+3}\right)$ D) $\left(\frac{1\times -6-3\times 2}{1-3},\frac{1\times 3-3\times -1}{1-3}\right)$

2. Find $\int \frac{1}{\sqrt{16-9x^2}}\,dx$.

A) $\sin^{-1}\left(\frac{x}{4}\right)+C$ B) $\sin^{-1}\left(\frac{3x}{4}\right)+C$

C) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right)+C$ D) $\frac{1}{4}\sin^{-1}\left(\frac{3x}{4}\right)+C$

3. What is the natural domain of $f(x)=\log_e(\cos^{-1}x)$?

A) $0<x\le 1$ B) $-1<x\le 1$

C) $0\le x<1$ D) $-1\le x<1$

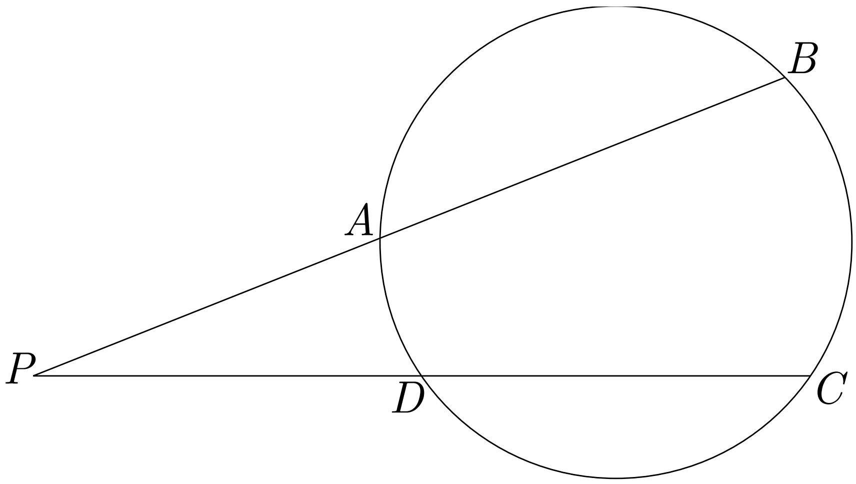
4. Which of the following is a valid solution to the differential equation $\frac{dP}{dt}=-k(P-M)$, where $k$ and $M$ are constants?

A) $P=Me^{kt}$ B) $P=Me^{-kt}$

C) $P=M+e^{kt}$ D) $P=M+e^{-kt}$

(Section A continues on the next page...)

5.

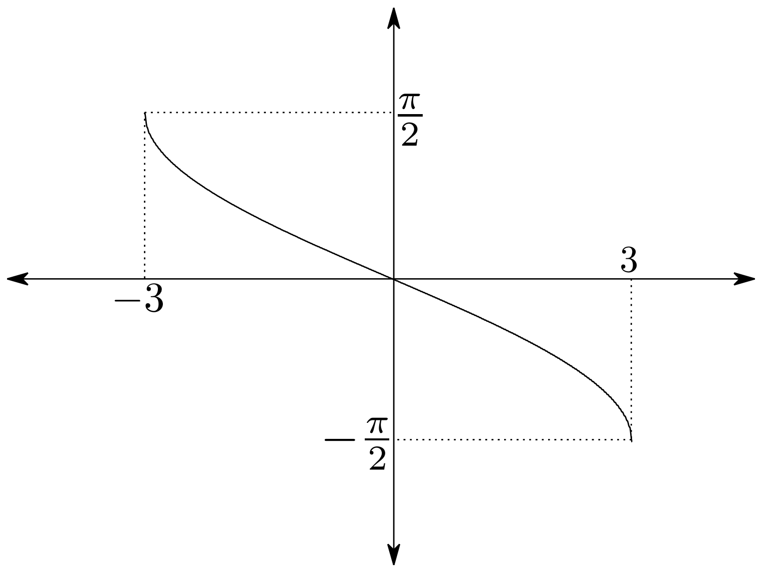



Which is the correct relation for the intervals in the diagram above?

A) $PA \times AB = PD \times DC$ B) $PA \times DC = PD \times AB$

C) $PA \times PB = PD \times PC$ D) $PA \times PC = PD \times PB$

6. Which of the following is the correct function for the graph below?




A) $y = -\sin^{-1}\left(\frac{x}{3}\right)$ B) $y = -\cos^{-1}\left(\frac{x}{3}\right)$

C) $y = -\sin^{-1}\left(3x\right)$ D) $y = -\cos^{-1}\left(3x\right)$

(Section A continues on the next page...)

7. For which function are the values $x = 2$ and $x = 4$ suitable starting values for estimating a zero using the method of "halving the interval"?

A) $f(x) = x - 5$

B) $f(x) = x^2 - 5$

C) $f(x) = x^3 - 5$

D) $f(x) = x^4 - 5$

8. The velocity $v$ of a particle moving in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where $x$ is the displacement from a fixed point. Given that the particle is in simple harmonic motion, what is the centre of motion?

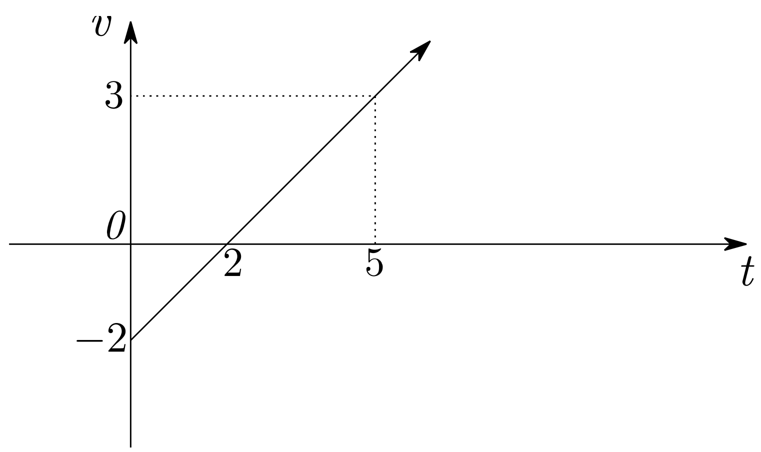
A) $x = -2$

B) $x = -1$

C) $x = 1$

D) $x = 2$

9. A particle moving on a straight line is depicted by the following velocity-time graph. What distance does it travel in the first 5 seconds of motion?




A) 2.5 units

B) 3 units

C) 5 units

D) 6.5 units

(Section A continues on the next page...)

10. Given that $\alpha$, $\beta$ and $\gamma$ are the roots of $x^3 - 3x^2 - 5x + 6 = 0$, what is the value of $\alpha^2 + \beta^2 + \gamma^2$?

A) -1

B) 4

C) 14

D) 19

*(End of Section A. Question 11 begins on the next page.)*

**Section B (60 marks)**

**Question 11** (Begin and label a new booklet.) **(15 marks)**

a) Find the acute angle between the lines $x-2y+1=0$ and $y=5x-4$. Give your answer in radians correct to two decimal places. [2]

b) Solve the inequality $\dfrac{3x+2}{x-2}>1$. [3]

c) Use Newton's method to find a better approximation to the root of $\log_e x-\sin x=0$, given that the root is near $x=2$. ($x$ is a radian)
Give your answer to 2 decimal places. [3]

d) Let $P(x)=(x+1)(x-3)Q(x)+a(x+1)+b$, where $Q(x)$ is a polynomial and $a$ and $b$ are real numbers.

When $P(x)$ is divided by $(x+1)$, the remainder is $-11$.
When $P(x)$ is divided by $(x-3)$, the remainder is 1.

i) What is the value of $b$? [1]

ii) What is the remainder when $P(x)$ is divided by $(x+1)(x-3)$? [2]

e) Prove by mathematical induction that [4]

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\ldots+\frac{1}{4n^2-1}=\frac{n}{2n+1}$$

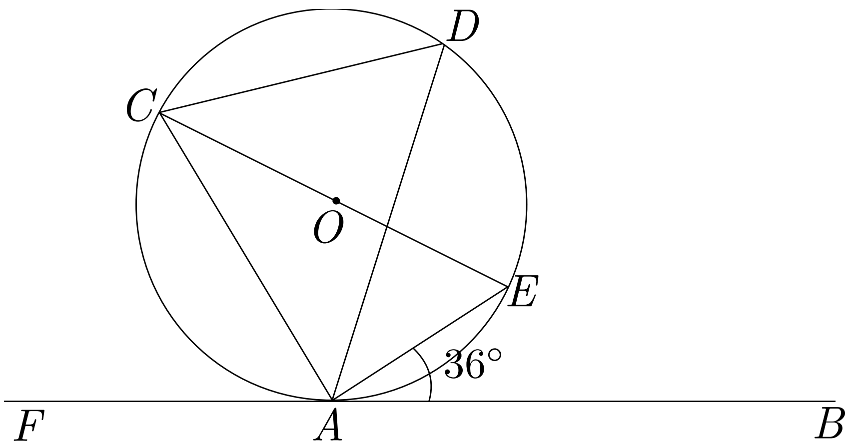
for positive integers $n$.

*(End of Question 11.)*

**Question 12** (Begin and label a new booklet.) **(15 marks)**

a) Evaluate $\int_0^{\frac{\pi}{8}} \cos^2 x \, dx$ [2]

b) $FB$ is a tangent touching a circle at $A$. $CE$ is the diameter, $O$ is the centre and $D$ lies on the circumference. $\angle BAE = 36°$.




i) Find the size of $\angle ACE$, giving reasons. [1]

ii) Find the size of $\angle ADC$, giving reasons. [2]

c) i) Show that the equation of tangent at point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ is given by $px - y - ap^2 = 0$. [2]

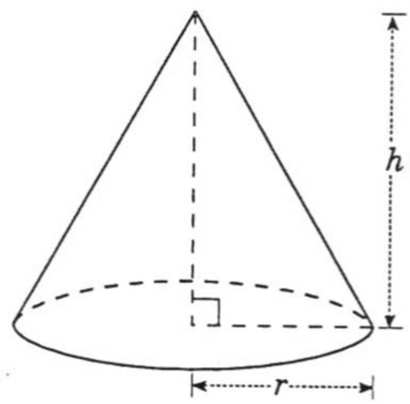
ii) $S$ is the focus of the parabola and $T$ is the point of intersection of the tangent and the y-axis.

Prove that $SP = ST$. [2]

iii) Hence show that $\angle SPT$ is equal to the acute angle between the tangent and the line through $P$ parallel to the axis of the parabola. [2]

*(Question 12 continues on the next page...)*

d) Sand is poured at a rate of 2 cubic metres per minute. It forms a conical pile, with the angle at the apex of the cone equal to $60°$. The height of the pile is $h$ metres, and the radius of the base is $r$ metres.




i) Show that $r = \dfrac{h}{\sqrt{3}}$. [1]

ii) Show that $V$, the volume of the pile, is given by $V = \dfrac{\pi h^3}{9}$. [1]

iii) Hence find the rate at which the height of the pile is increasing when the height of the pile is 3 metres. [2]

*(End of Question 12.)*

**Question 13** (Begin and label a new booklet.) **(15 marks)**

a) i) Find $\frac{d}{dx}(x\tan^{-1}x)$ [1]

ii) Hence evaluate $\int_0^1 \tan^{-1}x \, dx$ [2]

b) A particle is moving on a straight line with its displacement described by $x = \cos 2t + \sin 2t$, where $t$ is in seconds.

i) Show that the particle is in simple harmonic motion, namely its acceleration has the form $\ddot{x} = -n^2 x$. [2]

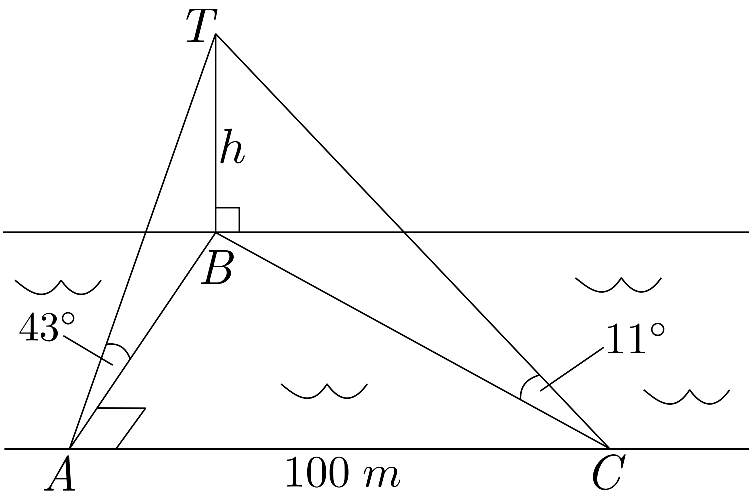
ii) State the period of motion. [1]

iii) By expressing $\cos 2t + \sin 2t$ in the form $R\cos(2t - \alpha)$, find the amplitude of the motion. [2]

iv) Hence or otherwise find the value of $t$ at the first moment that the particle is stationary. [2]

*(Question 13 continues on the next page...)*

c) A tree $BT$ is observed directly across a creek with an angle of elevation of $43°$ from the ground. After walking 100 metres along the bank from $A$ to $C$, the same tree now has an angle of elevation of $11°$.




i) Show that the creek has a width of $\dfrac{h}{\tan 43°}$. [1]

ii) Find the height of the tree to 2 decimal places. [4]

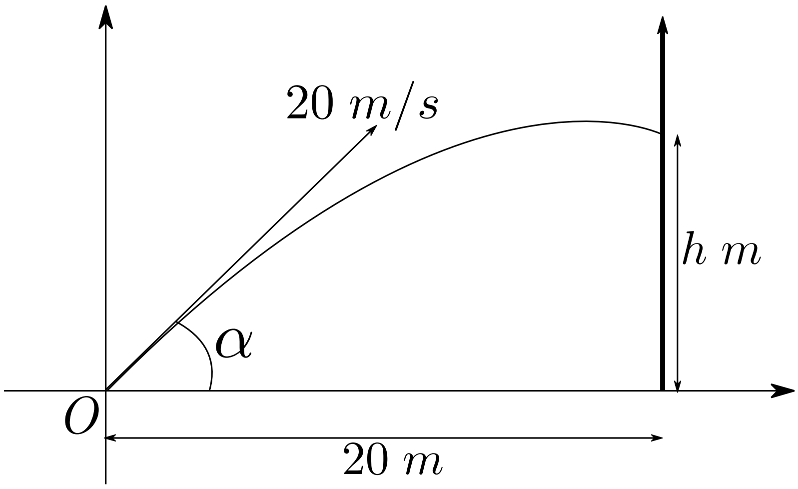
*(End of Question 13.)*

**Question 14** (Begin and label a new booklet.) **(15 marks)**

a) By using the substitution $u = x^2 - 5$, or otherwise, find $\int \frac{x^3}{\sqrt{x^2-5}}\,dx$. [3]

b) Show that $\frac{1-\tan x}{1+\tan x} = \frac{1-2\sin x\cos x}{\cos^2 x - \sin^2 x}$ [3]

c) A cricketer hits the ball from ground level with a speed of 20 m/s and an angle of elevation of $\alpha$. It flies towards a high wall 20 metres away.
Take $g = 10\ m/s^2$




i) Given that the horizontal and vertical displacements at time $t$ are, respectively (do not need to derive these):

$$x = 20t\cos\alpha$$
$$y = -5t^2 + 20t\sin\alpha$$

Show that the value of $h$, the height up the wall at which the ball will collide, is given by $h = -5\sec^2\alpha + 20\tan\alpha$. [2]

ii) Show that the maximum value of $h$ is obtained when $\tan\alpha = 2$. [2]

iii) Assuming $\tan\alpha = 2$, find the speed at which the ball will hit the wall. [2]

*(Question 14 continues on the next page...)*

d) i) State any discontinuities, if any exist, of the function $f(x) = \dfrac{1+e^x}{1-e^x}$. [1]

ii) It is given that $f'(x) = \dfrac{2e^x}{(1-e^x)^2}$.

Sketch $y = f(x)$, showing all important features. [2]

*End of exam.*

# 2015

## Ext I Trial Solutions

Section A. Q1. C  Q2. C  Q3. D  Q4. D  Q5. C

Q6. A  Q7. B  Q8. C  Q9. D  Q10. D

Q11. a) $x - 2y + 1 = 0$

$2y = x + 1$

$y = \frac{x}{2} + \frac{1}{2}$

$m_1 = \frac{1}{2}$

$y = 5x - 4$

$m_2 = 5$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{\frac{1}{2} - 5}{1 + \frac{5}{2}}\right|$ ✓1

$= \frac{9}{7}$

$\theta = 52.125^\circ \ldots$

$\doteqdot 0.91^c$ to 2d.p. ✓1

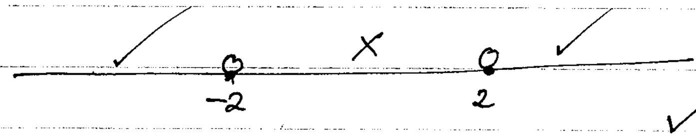
b). $\frac{3x+2}{x-2} > 1$

Disc: $x = 2$.

Solutions: $3x + 2 = x - 2$

$2x = -4$

$x = -2$ ✓1




✓1

$\therefore x < -2, \ x > 2$. ✓1

c) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ ✓1

$f(2) \approx -0.21615 \ldots$

$f'(x) = \frac{1}{x} - \cos x$

$f'(2) = 0.91614 \ldots$ ✓1

$x_2 = 2 - \frac{-0.21615}{0.91614}$

$= 2.2359 \ldots$

$\approx 2.24$ to 2d.p. ✓1

d) i) $P(-1) = -11$ by remainder theorem.

$\therefore -11 = 0 + 0 + b$

$b = -11$ ✓1

ii) $P(3) = 1$

$P(3) = 0 + 4a + b$

$1 = 4a + b$ ✓1

$1 = 4a - 11$

$4a = 12$  $a = 3$

Remainder is $3(x+1) - 11 = 3x - 8$ ✓1

Q11e) let $n=1$, $\frac{1}{3} = \frac{1}{2\times1+1}$

$\frac{1}{3} = \frac{1}{4\times1^2-1}$ true for $n=1$ ✓ 1

Assume for $n=k$, $\frac{1}{3} + \frac{1}{15} + \ldots + \frac{1}{4k^2-1} = \frac{k}{2k+1}$

RTP for $n=k+1$ $\frac{1}{3} + \frac{1}{15} + \ldots + \frac{1}{4k^2-1} + \frac{1}{4(k+1)^2-1} = \frac{k+1}{2(k+1)+1}$

$$LHS = \frac{k}{2k+1} + \frac{1}{4(k+1)^2-1}$$ by assumption ✓ 1

$$= \frac{k}{2k+1} + \frac{1}{(2k+2-1)(2k+2+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$ $RHS = \frac{k+1}{2(k+1)+1}$

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$ $= \frac{k+1}{2k+3}$

$$= \frac{k+1}{2k+3}$$ 2 ✓✓

$\therefore LHS = RHS$

$\therefore$ Statement true for $n=1, 2, \ldots$ all positive integers $n$ by mathematical induction.

Q12. a) $\int_0^{\frac{\pi}{8}} \cos^2 x \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{8}} 1+\cos 2x \, dx$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{8}}$

$= \frac{1}{2}\left(\frac{\pi}{8} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\right) - \frac{1}{2}(0)$

$= \frac{\pi}{16} + \frac{1}{2\sqrt{2}}$

b). i) $\angle ACE = \angle EAB$ —(angle in alternate segment)

$= 36°$

ii) $\angle CAE = 90°$ —(angle in semicircle)

$\therefore \angle CEA = 180 - 36 - 90$ —(angle sum of triangle)

$= 54°$

$\angle CDA = \angle CEA$ —(angles standing on same arc are equal)

$= 54°$

c) i) $\frac{dy}{dx} = \frac{dy/dp}{dx/dp}$

$= \frac{2ap}{2a}$

$= p$

Eqn tangent: $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

$px - y - ap^2 = 0$

ii) $S: (0, a)$

$T:$ $x = 0,$ $-y - ap^2 = 0$ $y = -ap^2$

$\therefore T(0, -ap^2)$

$SP = \sqrt{(2ap)^2 + (ap^2 - a)^2}$

$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$

$= \sqrt{a^2(p^4 + 2p^2 + 1)}$

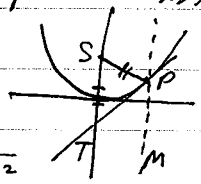
$= \sqrt{a^2(p^2+1)^2}$

$SP = a(p^2+1)$, $SP > 0$.

$ST = |a - -ap^2|$

$= ap^2 + a$

$\therefore SP = ST$




iii) $\angle SPT = \angle STP$

—(base ∠s of isosceles △ are equal)

$\angle TPM = \angle STP$ - (alternate ∠s equal, ST ∥ PM)

$\therefore \angle SPT = \angle TPM$

as required.

Q12d) i) $\tan 30 = \frac{r}{h}$

$r = \frac{h}{\sqrt{3}}$ ✓ 1

ii) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times \left(\frac{h^2}{3}\right) \times h$

$= \frac{\pi h^3}{9}$ ✓ 1

iii) $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$V = \frac{\pi h^3}{9}$

$\frac{dV}{dh} = \frac{3\pi h^2}{9}$

$= \frac{\pi h^2}{3}$ ✓ 1

$\frac{dV}{dt} = 2,$

at $h = 3,$ $\frac{dV}{dh} = \frac{\pi \times 3^2}{3}$

$= 3\pi$

$2 = 3\pi \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{2}{3\pi}$ m/min. ✓ 1

Q13 a) i) $\frac{d}{dx}(x\tan^{-1}x)$

$= \tan^{-1}x + \frac{x}{1+x^2}$ ✓

ii) $\therefore \int \tan^{-1}x + \frac{x}{1+x^2}\,dx = x\tan^{-1}x + c$

$\int \tan^{-1}x\,dx + \int \frac{x}{1+x^2}\,dx = x\tan^{-1}x + c$

$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$

$\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\log_e(1+x^2) + c$ ✓

$\int_0^1 \tan^{-1}x = \left[x\tan^{-1}x - \frac{1}{2}\log_e(1+x^2)\right]_0^1$

$= (1\tan^{-1}1 - \frac{1}{2}\log_e(2)) - (0 - \log_e 1)$ ✓

$= \frac{\pi}{4} - \frac{1}{2}\log_e 2.$ ✓

b) i) $x = \cos 2t + \sin 2t$

$\dot{x} = -2\sin 2t + 2\cos 2t$ ✓

$\ddot{x} = -4\cos 2t - 4\sin 2t$

$= -4(\cos 2t + \sin 2t)$

$= -4x$ ✓

$\therefore$ SHM.

ii) $n^2 = 4,\ n = 2,\ T = \frac{2\pi}{2} = \pi$ sec. ✓

iii) $\cos 2t + \sin 2t = R\cos(2t - \alpha)$

LHS $= \sqrt{2}(\frac{1}{\sqrt{2}}\cos 2t + \frac{1}{\sqrt{2}}\sin 2t)$

$= \sqrt{2}(\cos 2t\cos\frac{\pi}{4} + \sin 2t\sin\frac{\pi}{4})$

$= \sqrt{2}\cos(2t - \frac{\pi}{4})$ ✓

Amplitude : $\sqrt{2}$ units ✓

iv) $\dot{x} = -2\sqrt{2}\sin(2t - \frac{\pi}{4})$ ✓

$\dot{x} = 0$: $\sin(2t - \frac{\pi}{4}) = 0$

$2t - \frac{\pi}{4} = 0, \pi, \ldots$

$2t = \frac{\pi}{4}, \frac{5\pi}{4} \ldots$ $t = \frac{\pi}{8}$ is the first time $\dot{x} = 0$. ✓

c) i) $\frac{h}{BA} = \tan 43$

$\therefore BA = \frac{h}{\tan 43°}$ ✓1

ii) By Pythagoras: $BC = \frac{h}{\tan 11°}$ ✓1

$AC^2 + AB^2 = BC^2$

$\therefore 10000 + \frac{h^2}{\tan^2 43} = \frac{h^2}{\tan^2 11}$ ✓1

$\therefore 10000 = h^2\left(\frac{1}{\tan^2 11} - \frac{1}{\tan^2 43}\right)$

$10000 =$ ~~$0.445382\ h^2$~~ $25.31646..h^2$ ✓1

$h^2 = 394.9999...$

$h \doteqdot 19.87$ m. ✓1

Q14 a) $\int \frac{x^3}{\sqrt{x^2-5}}\,dx$ $u = x^2 - 5$, $x^2 = u + 5$

$du = 2x\,dx$

$= \frac{1}{2}\int \frac{2x \cdot x^2}{\sqrt{x^2-5}}\,dx$ ✓

$= \frac{1}{2}\int \frac{u+5}{\sqrt{u}}\,du$

$= \frac{1}{2}\int u^{\frac{1}{2}} + 5u^{-\frac{1}{2}}\,du$ ✓

$= \frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{5}{2}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}\frac{1}{2}u^{\frac{3}{2}} + \frac{5}{2}\cdot\frac{2}{1}u^{\frac{1}{2}} + c$

$= \frac{1}{3}u^{\frac{3}{2}} + 5u^{\frac{1}{2}} + c$ ✓

$= \frac{1}{3}(x^2-5)^{\frac{3}{2}} + 5(x^2-5)^{\frac{1}{2}} + c$

b) ~~LHS = $\frac{1 - 2\sin x\cos x}{\cos^2 x - \sin^2 x}$ =~~

~~RHS~~ $= \frac{1-\tan x}{1+\tan x}$

LHS

$= \frac{\left(1 - \frac{\sin x}{\cos x}\right)\times \cos x}{\left(1 + \frac{\sin x}{\cos x}\right)\times\cos x}$

$= \frac{\cos x - \sin x}{\cos x + \sin x}$ ✓

$= \frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x}$

$= \frac{\cos^2 x + \sin^2 x - 2\sin x\cos x}{\cos^2 x - \sin^2 x}$ ✓

$= \frac{1 - 2\sin x\cos x}{\cos^2 x - \sin^2 x}$ as required

= ~~LHS~~ RHS ✓

e) i) $x = 20t\cos\alpha$

$y = -5t^2 + 20t\sin\alpha$

$x = 20, \quad 20 = 20t\cos\alpha$

$t = \frac{1}{\cos\alpha}$ ✓

$\therefore h = -5 \cdot \frac{1}{\cos^2\alpha} + 20 \cdot \frac{\sin\alpha}{\cos\alpha}$

$h = -5\sec^2\alpha + 20\tan\alpha$ ✓

ii) $h = -5(\tan^2\alpha + 1) + 20\tan\alpha$

$h = -5\tan^2\alpha + 20\tan\alpha - 5$ ✓

let $\tan\alpha = u$

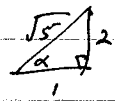
$h = -5u^2 + 20u - 5$

max is at axis of symm: $u = \frac{-b}{2a} = \frac{-20}{-10}$

$= 2$ ✓

max $h$ is at $\tan\alpha = 2$

iii) $\tan\alpha = 2$




$\sin\alpha = \frac{2}{\sqrt{5}}$

$\cos\alpha = \frac{1}{\sqrt{5}}$

$x = 20, \quad t = \frac{1}{\cos\alpha} = \sqrt{5}$

$\therefore \dot{x} = 20\cos\alpha$

$= \frac{20}{\sqrt{5}}$ ✓

$\dot{y} = -10t + 20\sin\alpha$

$= -10\sqrt{5} + \frac{40}{\sqrt{5}} = -10\sqrt{5} + 8\sqrt{5} = -2\sqrt{5}$

$\text{speed}^2 = \dot{x}^2 + \dot{y}^2$

$= \left(\frac{20}{\sqrt{5}}\right)^2 + (-2\sqrt{5})^2$

$= 80 + 20$

$= 100$

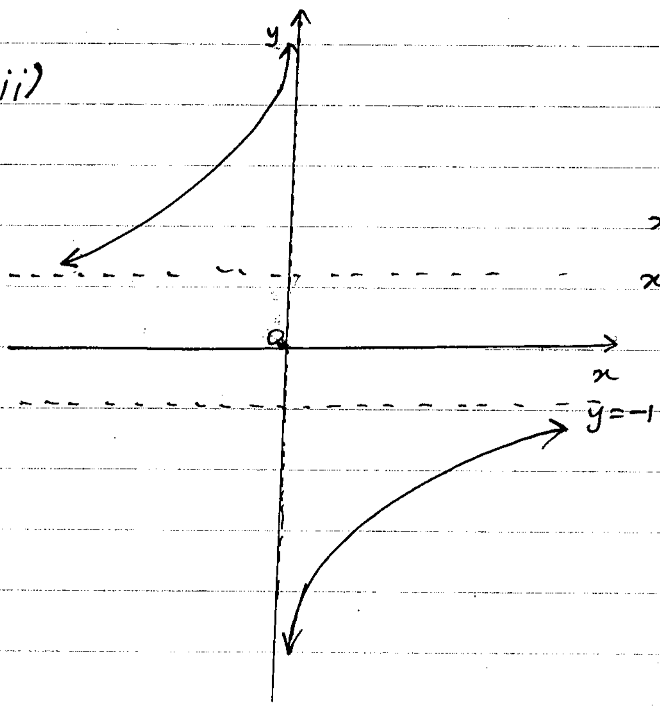
speed $= 10$ m/s ✓

d) i) $1-e^{x}=0$

$e^{x}=1$

$x=0.$ ✓ 1

ii)




$x=0.1,\ y\to-\infty$

$x=-0.1,\ y\to+\infty$

$x\to+\infty,\ y\to-1$

$x\to-\infty,\ y\to1$

vertical asymptote ✓ 1

horizontal asymptotes ✓ 1